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Exact Results on Potts Model Partition Functions in a Generalized External Field and Weighted-Set Graph Colorings

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Abstract We present exact results on the partition function of the q-state Potts model on various families of graphs G in a generalized external magnetic field that favors or disfavors spin values in a subset $I_s = \{1, \dots, s\}$ of the total set of possible spin values, $Z(G, q, s, v, w)$, where v and w are temperature- and field-dependent Boltzmann variables. We remark on differences in thermodynamic behavior between our model with a generalized external magnetic field and the Potts model with a conventional magnetic field that favors or disfavors a single spin value. Exact results are also given for the interesting special case of the zero-temperature Potts antiferromagnet, corresponding to a set-weighted chromatic polynomial $Ph(G, q, s, w)$ that counts the number of colorings of the vertices of G subject to the condition that colors of adjacent vertices are different, with a weighting w that favors or disfavors colors in the interval I_s. We derive powerful new upper and lower bounds on $Z(G, q, s, v, w)$ for the ferromagnetic case in terms of zero-field Potts partition functions with certain transformed arguments. We also prove general inequalities for $Z(G, q, s, v, w)$ on different families of tree graphs. As part of our analysis, we elucidate how the field-dependent Potts partition function and weighted-set chromatic polynomial distinguish, respectively, between Tutte-equivalent and chromatically equivalent pairs of graphs.

Keywords Potts model in an external field · Weighted-set graph colorings

1 Introduction

In this paper we continue our study of the q-state Potts model in a generalized external magnetic field that favors or disfavors a certain subset of spin values in the interval $I_s = \{1, \dots, s\}$, on various families of graphs G [1, 4]. We denote a graph $G = (V, E)$ by

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its vertex set V and its edge (= bond) set E. The numbers of vertices, edges, and connected components of G are denoted, respectively, by $n(G) \equiv n$, $e(G)$, and $k(G)$. In thermal equilibrium at temperature T, the partition function for the Potts model on the graph G in this field is given by $Z = \sum_{\{\sigma_i\}} e^{-\beta\mathcal{H}}$ with the Hamiltonian

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \delta_{\sigma_i,\sigma_j} - \sum_{p=1}^{q} H_p \sum_{\ell} \delta_{\sigma_\ell, p}, \tag{1.1}$$

where i, j, ℓ label vertices of G, σ_i are classical spin variables on these vertices, taking values in the set $I_q = \{1, \ldots, q\}$, $\beta = (k_B T)^{-1}$, $\langle ij \rangle$ denote pairs of adjacent vertices, J is the spin-spin interaction constant, and

$$H_p = \begin{cases} H & \text{for } 1 \le p \le s, \\ 0 & \text{for } s+1 \le p \le q. \end{cases} \tag{1.2}$$

Thus, for positive (negative) H, the Hamiltonian favors (disfavors) spin values in the interval I_s. This is a generalization of a conventional magnetic field, which would favor or disfavor one particular spin value. We denote $I_s^{\perp}$ as the orthogonal complement of I_s in I_q, i.e., $I_s^{\perp} = \{s+1, \ldots, q\}$, and we use the notation

$$K = \beta J, \qquad h = \beta H, \qquad y = e^K, \qquad v = y - 1, \qquad w = e^h. \tag{1.3}$$

The physical ranges of v are $v \ge 0$ for the Potts ferromagnet, and $-1 \le v \le 0$ for the Potts antiferromagnet.

It is very useful to have a general graph-theoretic formula for Z that does not make any explicit reference to the spins σ_i or the summation over spin configurations, but instead expresses this function as a sum of terms arising from the spanning subgraphs $G' \subseteq G$. This formula was derived and analyzed in Refs. [3, 4] and is

$$Z(G, q, s, v, w) = \sum_{G' \subseteq G} v^{e(G')} \prod_{i=1}^{k(G')} u_{n(G'_i)} \tag{1.4}$$

where

$$u_m = q - s + s w^m = q + s(w^m - 1). \tag{1.5}$$

This generalizes a spanning subgraph formula for Z in the case $s = 1$ due to F.Y. Wu [5, 6]. In the special case $H = 0$, (1.4) reduces to the cluster formula for the zero-field Potts model partition function [7, 8], denoted $Z(G, q, v)$, namely

$$Z(G, q, v) = \sum_{G' \subseteq G} v^{e(G')} q^{k(G')}. \tag{1.6}$$

The original definition of the Potts model, (1.1), requires q to be in the set of positive integers $\mathbb{N}_+$ and s to be a non-negative integer. These restrictions are removed by (1.4). Furthermore, (1.4) shows that Z is a polynomial in the variables q, s, v, and w, hence our notation $Z(G, q, s, v, w)$. If two graphs G_1 and G_2 are disjoint, then $Z(G_1 \cup G_2) = Z(G_1)Z(G_2)$ so, without loss of generality, we will usually restrict to connected G (although (1.4) leads to consideration of disconnected spanning subgraphs G').

An important special case is the zero-temperature antiferromagnet, $K = -\infty$, i.e., $v = -1$, and we denote

$$Ph(G, q, s, w) \equiv Z(G, q, s, -1, w). \tag{1.7}$$

In this case the only contributions to Z are those such that no two adjacent spins have the same value. Thus, $Ph(G, q, s, w)$ counts the number of proper q-colorings of the vertices of G with a vertex weighting that either disfavors (for $0 \le w < 1$) or favors (for $w > 1$) colors in the interval I_s. Here, a proper q-coloring is defined as an assignment of q colors to the vertices of a graph G subject to the condition that no two adjacent vertices have the same color. We have denoted these coloring problems as DFSCP and FSCP for disfavored or favored weighted-set graph vertex coloring problems [4]. The associated set-weighted chromatic polynomial constitutes a generalization of the conventional (unweighted) chromatic polynomial, which counts the number of proper q-colorings of a graph G. Recent reviews of chromatic polynomials include [9–11].

As is evident from the discussion above, the model defined by (1.1) with (1.2) is of interest both in the context of statistical mechanics and in the context of mathematical graph theory. It also has an application to certain frequency allocation problems in electrical engineering [4].

2 Some Basic Properties of $Z(G, q, s, w, v)$ and $Ph(G, q, s, w)$

In this section we discuss some basic results about $Z(G, q, s, v, w)$ and $Ph(G, q, s, w)$ that will be needed in our work. Applying the factorization

$$w^m - 1 = (w - 1) \sum_{j=0}^{m-1} w^j \tag{2.1}$$

in (1.4) with $m = n(G'_i)$, one sees that the variable s enters in $Z(G, q, s, v, w)$, and $Ph(G, q, s, w)$ only in the combination

$$t = s(w - 1). \tag{2.2}$$

Since $I_s \subseteq I_q$, whence $0 \le s \le q$, and since $w \ge 0$ for any real external field H, it follows that

$$u_m = q + s(w^m - 1) \ge 0. \tag{2.3}$$

Therefore, for the ferromagnetic case $v \ge 0$, each term in the sum over spanning subgraphs in (1.4) is nonnegative. For a given spanning subgraph $G' \subseteq G$, consisting of a sum of $k(G')$ connected components G'_i, where $i = 1, \dots, k(G')$, the contribution to $Z(G, q, s, v, w)$ in (1.4) is the number of spanning subgraphs G' of a particular topology, $N_{G'}$, times $v^{e(G')} \prod_{i=1}^{k(G')} u_{n(G'_i)}$, which has the generic form

$$N_{G'} v^{e(G')} \prod_{j=1}^{k(G')} u_{n(G'_j)}. \tag{2.4}$$

Here

$$\sum_{i=1}^{k(G')} n(G'_i) = n. \tag{2.5}$$

Since some of the components G'_i and G'_j may have the same number of vertices, $n(G'_i) = n(G'_j)$, the product in (2.4) can also be written as $\prod_j (u_{r_j})^{p_j}$, where r_j takes on certain values in the set $\{1,\dots,n\}$ and the exponents p_j are integers taking on certain values in the set $\{1,\dots,k(G')\}$. As a consequence of (2.5), these satisfy the relation

$$\sum_j p_j r_j = n. \tag{2.6}$$

Note that u_m satisfies the identity

$$u_m(q,s,w) = w^m u_m(q,q-s,w^{-1}) \tag{2.7}$$

where we have written u_m as a function of its three arguments q, s, w. A given spanning subgraph G' corresponds to a partition of the total set of vertices depending on which edges are present and which are absent. The sum of the coefficients $N_{G'}$ of the various terms $N_{G'} \prod_j (u_{r_j})^{p_j}$ that multiply a given power $v^{e(G')}$ in (1.4) is $\binom{e(G)}{e(G')}$ since this is the number of ways of choosing $e(G')$ edges out of a total of $e(G)$ edges. These satisfy the relation

$$\sum_{e(G')=0}^{e(G)} \binom{e(G)}{e(G')} = 2^{e(G)}. \tag{2.8}$$

This reflects the fact that there are $2^{e(G)}$ spanning subgraphs of G, as follows from the property that these are classified by choosing whether each edge is present or absent, and there are $2^{e(G)}$ such choices. In mathematical graph theory, a loop is defined as an edge that connects a vertex to itself and a cycle is a closed circuit along the edges of G. In the following we restrict to loopless graphs. For any such n-vertex graph G, the terms in $Z(G,q,s,v,w)$ proportional to v^0, v^1, and $v^{e(G)}$ can be given in general, as

$$Z(G,q,s,v,w) = u_1^n + e(G) v u_2 u_1^{n-2} + \cdots + v^{e(G)} u_n. \tag{2.9}$$

The partition function $Z(G,q,s,v,w)$ satisfies the following identities [1–4]

$$Z(G,q,s,v,1) = Z(G,q,0,v,w) = Z(G,q,v), \tag{2.10}$$

(where, as above, $Z(G,q,v)$ is the zero-field Potts partition function),

$$Z(G,q,s,v,w) = w^n Z(G,q,q-s,v,w^{-1}), \tag{2.11}$$

(c.f. (2.7)) and

$$Z(G,q,q,v,w) = w^n Z(G,q,v). \tag{2.12}$$

Setting $v=-1$ in these identities yields the corresponding relations for $Ph(G,q,s,w)$; for example, (2.11) yields

$$Ph(G,q,s,w) = w^n Ph(G,q,q-s,w^{-1}). \tag{2.13}$$

There are a number of equivalent ways of writing $Z(G,q,s,v,w)$ as sums of powers of a given variable with coefficients depending on the rest of the variables in the set $\{q,s,v,w\}$. The basic spanning subgraph formula (1.4) is a sum of powers of v. A second convenient form in which to express $Z(G,q,s,v,w)$ is as a sum of powers of w with coefficients, denoted as $\beta_{Z,G,j}(q,s,v)$, which are polynomials in q, s, and v:

$$Z(G,q,s,v,w)=\sum_{j=0}^{n}\beta_{Z,G,j}(q,s,v)w^j. \tag{2.14}$$

The symmetry (2.11) implies the following relation among the coefficients:

$$\beta_{Z,G,j}(q,s,v)=\beta_{Z,G,n-j}(q,q-s,v) \quad \text{for } 0\le j\le n. \tag{2.15}$$

For the special case $v=-1$, we write

$$Ph(G,q,s,w)=\sum_{j=0}^{n}\beta_{G,j}(q,s)w^j, \tag{2.16}$$

where

$$\beta_{G,j}(q,s)\equiv\beta_{Z,G,j}(q,s,-1). \tag{2.17}$$

From (2.15), we have

$$\beta_{G,j}(q,s)=\beta_{G,n-j}(q,q-s) \quad \text{for } 0\le j\le n. \tag{2.18}$$

We have proved further that [4]

$$\beta_{Z,G,n}(q,s,v)=Z(G,s,v) \tag{2.19}$$

and

$$\beta_{Z,G,0}(q,s,v)=Z(G,q-s,v), \tag{2.20}$$

so that for $v=-1$, $\beta_{G,0}(q,s)=P(G,q-s)$ and $\beta_{G,n}(q,s)=P(G,s)$. Various general factorization results were also given in Ref. [4] for these coefficients $\beta_{Z,G,j}(q,s,v)$ and $\beta_{G,j}(q,s)$, including the following:

$$\text{For } 1\le j\le n, \quad \beta_{Z,G,j}(q,s,v) \text{ and } \beta_{G,j}(q,s) \text{ contain a factor of } s. \tag{2.21}$$

$$\text{For } 0\le j\le n-1, \quad \beta_{Z,G,j}(q,s,v) \text{ and } \beta_{G,j}(q,s) \text{ contain a factor } (q-s). \tag{2.22}$$

The minimum number of colors needed for a proper q-coloring of a graph G is the chromatic number, $\chi(G)$. A further factorization property is that

$$\beta_{G,n}(q,s) \text{ contains the factor } \prod_{j=0}^{\chi(G)-1}(s-j), \tag{2.23}$$

and

$$\beta_{G,0}(q,s) \text{ contains the factor } \prod_{j=0}^{\chi(G)-1}(q-s-j). \tag{2.24}$$

A third useful type of expression for $Z(G,q,s,v,w)$ is

$$Z(G,q,s,v,w)=\sum_{j=0}^{n}\alpha_{Z,G,n-j}(s,v,w)q^{n-j}. \tag{2.25}$$

With the notation

$$\alpha_{G,n-j}(s,w)\equiv\alpha_{Z,G,n-j}(s,-1,w), \tag{2.26}$$

we then have

$$Ph(G,q,s,w)=\sum_{j=0}^{n}\alpha_{G,n-j}(s,w)q^{n-j}. \tag{2.27}$$

This form is particularly convenient for comparisons with the conventional unweighted chromatic polynomial $P(G,q)=Ph(G,q,0,w)=Ph(G,q,s,1)$.

For a graph G, the number of linearly independent cycles, $c(G)$ (the cyclotomic number), satisfies the relation

$$c(G)=e(G)+k(G)-n(G). \tag{2.28}$$

A connected n-vertex graph with no cycles is a tree graph, T_n, while a general graph with no cycles, which can be disconnected, is called a forest graph. We denote a graph G with no cycles as G_{nc} and define

$$q'\equiv\frac{q}{s},\qquad v'\equiv\frac{v}{s}. \tag{2.29}$$

In Ref. [4] we proved that for such a cycle-free graph G_{nc},

$$Z(G_{nc},q,s,v,w)=s^nZ(G_{nc},q',1,v',w). \tag{2.30}$$

This relation allows us to obtain $Z(G_{nc},q,s,v,w)$ from $Z(G_{nc},q,1,v,w)$ for any cycle-free graph G_{nc}. In particular, all of the results for $Z(G,q,s,v,w)$ for various types of tree graphs calculated in Ref. [3] for $s=1$ can be used to obtain the analogous results for general s.

For a graph G, let us denote the graph obtained by deleting an edge $e\in E$ as $G-e$ and the graph obtained by deleting this edge and identifying the two vertices that had been connected by it as G/e. The Potts model partition function satisfies the deletion-contraction relation (DCR)

$$Z(G,q,v)=Z(G-e,q,v)+vZ(G/e,q,v), \tag{2.31}$$

and, setting $v=-1$, the chromatic polynomial thus satisfies the DCR

$$P(G,q)=P(G-e,q)-P(G/e,q). \tag{2.32}$$

However, as we showed in Ref. [4], in general, neither $Z(G,q,s,v,w)$ nor $Ph(G,q,s,w)$ satisfies the respective deletion-contraction relation, i.e., in general, $Z(G,q,s,w,v)$ is not equal to $Z(G-e,q,s,w,v)+vZ(G/e,q,s,w,v)$. The only cases where this deletion-contraction relation holds are for the values $s=0$, $w=1$, and $w=0$ where $Z(G,q,s,v,w)$ reduces to a zero-field Potts model partition function.

3 Upper and Lower Bounds on $Z(G,q,s,v,w)$ for $v \ge 0$

In this section we derive powerful new two-sided upper and lower bounds for the generalized field-dependent partition function of the ferromagnetic ($v \ge 0$) Potts model, $Z(G,q,s,v,w)$ on an arbitrary graph G in terms of the zero-field Potts model partition functions $Z(G,u_1,v)$ and $Z(G,u_1/w,v)$, where $u_1 = q + s(w-1)$ (c.f. (1.5)). These are especially useful because the zero-field Potts model partition function is considerably easier to calculate than $Z(G,q,s,v,w)$. Throughout this section, it is understood that $q \ge 0$, $0 \le s \le q$, and $v \ge 0$. The former two conditions are obvious for our present analysis, while the latter will often be indicated explicitly.

We first derive a lower bound for $Z(G,q,s,v,w)$ for the range $w \ge 1$. To begin, we observe that, from its definition in (1.5) and factorization property (2.1), u_m satisfies

$$\begin{aligned} u_m &= q + s(w^m - 1) = q + s(w-1)\sum_{j=0}^{m-1} w^j \\ &\ge q + s(w-1) = u_1 \quad \text{for } w \ge 1. \end{aligned} \tag{3.1}$$

Substituting this inequality into the expression for $Z(G,q,s,v,w)$ in (1.4) in terms of contributions from spanning subgraphs $G' \subseteq G$, we have, for the same conditions

$$\begin{aligned} Z(G,q,s,v,w) &= \sum_{G' \subseteq G} v^{e(G')} \prod_{i=1}^{k(G')} u_{n(G'_i)} \\ &\ge \sum_{G' \subseteq G} v^{e(G')} (u_1)^{k(G')} \quad \text{for } v \ge 0 \text{ and } w \ge 1. \end{aligned} \tag{3.2}$$

But the expression on the second line of (3.2) is just the zero-field Potts model partition function given in (1.6) with its argument q replaced by u_1, namely $Z(G,u_1,v)$. Hence, we have derived a lower bound on $Z(G,q,s,v,w)$:

$$Z(G,q,s,v,w) \ge Z(G,u_1,v) \quad \text{for } v \ge 0 \text{ and } w \ge 1. \tag{3.3}$$

For the interval $0 \le w \le 1$, the inequality (3.1) is reversed:

$$u_m \le u_1 \quad \text{for } 0 \le w \le 1, \tag{3.4}$$

and thus (3.2) is replaced by

$$Z(G,q,s,v,w) \le \sum_{G' \subseteq G} v^{e(G')} (u_1)^{k(G')} \quad \text{for } v \ge 0 \text{ and } 0 \le w \le 1. \tag{3.5}$$

Therefore, we obtain a second inequality, which is an upper bound:

$$Z(G,q,s,v,w) \le Z(G,u_1,v) \quad \text{for } v \ge 0 \text{ and } 0 \le w \le 1. \tag{3.6}$$

To derive two-sided inequalities, we make use of the symmetry relation (2.11), which maps the interval $w \ge 1$ to the interval $0 \le w \le 1$ and vice versa. Let us start with the case $w \ge 1$, for which we have proved the lower bound (3.3). Now, from the symmetry relation (2.11) we know that $Z(G,q,s,v,w) = w^n Z(G,q,\hat{s},v,\hat{w})$ where $\hat{s} \equiv q - s$ and

$\hat{w} \equiv w^{-1}$. Since $\hat{w} \in [0, 1]$, we can apply our upper bound (3.6) to $Z(G, q, \hat{s}, v, \hat{w})$, getting the inequality

$$Z(G, q, \hat{s}, v, \hat{w}) \le Z(G, \hat{u}_1, v), \tag{3.7}$$

where

$$\hat{u}_1 \equiv q + \hat{s}(\hat{w} - 1) = q + (q - s)(w^{-1} - 1) = \frac{u_1}{w}. \tag{3.8}$$

Combining (3.7) with (3.3), we derive the two-sided inequality

$$Z(G, u_1, v) \le Z(G, q, s, v, w) \le w^n Z\left(G, \frac{u_1}{w}, v\right) \quad \text{for } v \ge 0 \text{ and } w \ge 1. \tag{3.9}$$

For the interval $0 \le w \le 1$, by the same type of reasoning, we extend our upper bound (3.6) to the two-sided inequality

$$w^n Z\left(G, \frac{u_1}{w}, v\right) \le Z(G, q, s, v, w) \le Z(G, u_1, v) \quad \text{for } v \ge 0 \text{ and } 0 \le w \le 1. \tag{3.10}$$

As two-sided inequalities, these are powerful restrictions on the generalized field-dependent Potts model partition function in terms of zero-field Potts model partition functions with q replaced by u_1 and u_1/w.

We next prove some factorization properties of the upper and lower differences in these two-sided inequalities. First, if $w = 1$, then since $Z(G, q, s, v, 1) = Z(G, q, v)$ and $u_1 = q$, it follows that the two-sided inequalities (3.10) and (3.9) reduce to equalities, i.e., both the upper and lower differences vanish. Second, if $v = 0$, then the only contributions in the respective equations (1.4) and (1.6) are from the spanning subgraph with no edges (called the null graph, N_n), so $Z(G, q, s, 0, w) = (u_1)^n$, and $Z(G, q, 0) = q^n$, whence $Z(G, u_1, 0) = (u_1)^n$ and $w^n Z(G, u_1/w, 0) = (u_1)^n$. Hence, again, in this $v = 0$ case, the inequalities (3.10) and (3.9) reduce to equalities and the upper and lower differences vanish. Third, if $s = 0$, then $Z(G, q, 0, v, w) = Z(G, q, v)$ and $u_1 = q$, so that $Z(G, u_1, v) = Z(G, q, v)$. Hence, if $s = 0$, then the lower difference in (3.9) and the upper difference in (3.10) vanish. Fourth, if $w = 0$, then $Z(G, q, s, v, 0) = Z(G, q - s, v)$ and $u_1 = q - s$, so $Z(G, u_1, v) = Z(G, q - s, v)$; therefore, again, the lower difference in (3.9) and the upper difference in (3.10) vanish. Together, these four results prove that the difference

$$Z(G, q, s, v, w) - Z(G, u_1, v) \text{ contains the factor } w(w - 1)sv. \tag{3.11}$$

Fifth, if $s = q$, then $Z(G, q, q, v, w) = w^n Z(G, q, v)$ and $u_1 = qw$, so $w^n Z(G, u_1/w, v) = w^n Z(G, q, v)$. Hence, if $s = q$, then the upper difference in (3.9) and the lower difference in (3.10) vanish. Combining this with the first two results above, we have shown that

$$w^n Z\left(G, \frac{u_1}{w}, v\right) - Z(G, q, s, v, w) \text{ contains the factor } (w - 1)(q - s)v. \tag{3.12}$$

It is also useful to characterize the difference between the zero-field Potts model partition functions that constitute the upper and lower bounds in these two-sided inequalities (3.9) and (3.10). For an arbitrary graph G, we have

$$w^n Z\left(G, \frac{u_1}{w}, v\right) - Z(G, u_1, v) = \sum_{G' \subseteq G} v^{e(G')} (u_1)^{k(G')} \left[w^{n(G) - k(G')} - 1\right], \tag{3.13}$$

where G' is a spanning subgraph of G. Now the right-hand side of (3.13) is nonzero only if G has at least one edge, and, in this case, the only nonvanishing contributions have $n(G) - k(G') \geq 1$. It follows that

$$w^n Z\left(G, \frac{u_1}{w}, v\right) - Z(G, u_1, v) \text{ contains a factor } vu_1(w-1). \tag{3.14}$$

It is worthwhile to give some illustrations of these two-sided inequalities (3.9) and (3.10). We first do this for tree graphs. For any n-vertex tree graph T_n, if $w \geq 1$, then the inequality (3.9) reads

$$u_1(u_1+v)^{n-1} \leq Z(T_n, q, s, v, w) \leq u_1(u_1+wv)^{n-1} \quad \text{for } v \geq 0 \text{ and } w \geq 1, \tag{3.15}$$

where we have used $Z(T_n, q, v) = q(q+v)^{n-1}$. If $w \in [0, 1]$, then the inequality (3.10) reads

$$u_1(u_1+wv)^{n-1} \leq Z(T_n, q, s, v, w) \leq u_1(u_1+v)^{n-1} \quad \text{for } v \geq 0 \text{ and } 0 \leq w \leq 1. \tag{3.16}$$

(This example also shows how the apparent singularity at $w = 0$ arising from the u_1/w argument in $Z(G, u_1/w, v)$ on the left-hand side of the inequality (3.10) is removed by the w^n factor, yielding a nonsingular expression.) One gains further insight by calculating the differences between the polynomials that constitute the upper bound, the middle term, $Z(T_n, q, s, w, v)$, and the lower bound for various tree graphs. For the path graph L_2 and $w \geq 1$, the differences that enter in the two-sided inequality (3.15) are

$$u_1(u_1+wv) - Z(L_2, q, s, v, w) = (w-1)(q-s)v \geq 0 \tag{3.17}$$

and

$$Z(L_2, q, s, v, w) - u_1(u_1+v) = w(w-1)sv \geq 0. \tag{3.18}$$

For $w \in [0, 1]$ the differences that enter in (3.16) are obvious reversals of these, viz., $u_1(u_1 + v) - Z(L_2, q, s, v, w) = w(1-w)sv \geq 0$ and $Z(L_2, q, s, v, w) - u_1(u_1+wv) = (1-w)(q-s)v \geq 0$. For the path graph L_3 and $w \geq 1$, the differences in (3.15) are

$$u_1(u_1+wv)^2 - Z(L_3, q, s, v, w) = (w-1)(q-s)v\Big[2u_1 + v(w+1)\Big] \geq 0 \tag{3.19}$$

and

$$Z(L_3, q, s, v, w) - u_1(u_1+v)^2 = w(w-1)sv\Big[2u_1 + v(w+1)\Big] \geq 0, \tag{3.20}$$

and similarly for $w \in [0, 1]$.

Among n-vertex tree graphs, the star graph S_n has a particularly simple field-dependent Potts partition function, which was given in Ref. [4] and is derived by a direct evaluation of the general formula (1.4) (for any v):

$$\begin{aligned} Z(S_n, q, s, v, w) &= \sum_{j=0}^{n-1} \binom{n-1}{j} v^j u_{j+1} u_1^{n-1-j} \\ &= (q-s)\Big[q + s(w-1) + v\Big]^{n-1} + sw\Big[q + s(w-1) + wv\Big]^{n-1}. \end{aligned} \tag{3.21}$$

Here j is the number of edges in a given spanning subgraph G', and the numerical prefactor $\binom{n-1}{j}$ in the first line of (3.21) is the number of ways of choosing j edges out of the total number of edges, $n-1$, in S_n. For $v \ge 0$, substituting this result (3.21) into the two-sided inequalities (3.15) and (3.16), we can derive general formulas for the respective upper and lower differences. If $w \ge 1$ we find, for the lower difference in (3.15),

$$\begin{aligned} Z(S_n,q,s,v,w) - u_1(u_1+v)^{n-1} &= sw\Big[(u_1+wv)^{n-1} - (u_1+v)^{n-1}\Big] \\ &= sw\sum_{j=0}^{n-1}\binom{n-1}{j}(u_1)^{n-1-j}v^j(w^j-1) \\ &= sw(w-1)v\sum_{j=1}^{n-1}\binom{n-1}{j}(u_1)^{n-1-j}v^{j-1}\left[\sum_{\ell=0}^{j-1}w^\ell\right] \\ &\ge 0. \end{aligned} \tag{3.22}$$

In the same way, if $w \in [0,1]$, then the upper difference $u_1(u_1+v)^{n-1} - Z(S_n,q,s,v,w)$ in (3.16) is given by minus the right-hand side of (3.22). Similarly, if $w \ge 1$, then for the upper difference in (3.15) we calculate

$$\begin{aligned} u_1(u_1+wv)^{n-1} - Z(S_n,q,s,v,w) &= (q-s)\Big[(u_1+wv)^{n-1} - (u_1+v)^{n-1}\Big] \\ &= (q-s)(w-1)v\sum_{j=1}^{n-1}\binom{n-1}{j}(u_1)^{n-1-j}v^{j-1} \\ &\quad\times\left[\sum_{\ell=0}^{j-1}w^\ell\right] \ge 0. \end{aligned} \tag{3.23}$$

Again, if $w \in [0,1]$, then the lower difference $Z(S_n,q,s,v,w) - u_1(u_1+wv)^{n-1}$ in (3.16) is given by minus the right-hand side of (3.23).

For the circuit graph C_n, if $w \ge 1$, the inequality (3.15) reads $Z(C_n,u_1,v) \le Z(C_n,q,s,v,w) \le w^n Z(C_n,u_1/w,v)$. Using the fact that $Z(C_n,q,v) = (q+v)^n + (q-1)v^n$, we can write this explicitly as

$$(u_1+v)^n + (u_1-1)v^n \le Z(C_n,q,s,v,w) \le (u_1+wv)^n + (u_1-w)w^{n-1}v^n. \tag{3.24}$$

For C_2 (which has a double edge), the differences that enter in this two-sided inequality are

$$w^2 Z(C_2,u_1/w,v) - Z(C_2,q,s,v,w) = (q-s)(w-1)v(v+2) \ge 0 \tag{3.25}$$

and

$$Z(C_2,q,s,v,w) - Z(C_2,u_1,v) = w(w-1)sv(v+2) \ge 0. \tag{3.26}$$

Similar illustrations of the general inequalities (3.15) and (3.16) can be given for L_n and C_n with higher values of n and for other families of graphs.

For $w \ge 1$, we can prove a lower bound on u_m that is stronger than (3.1). To do this, we use the basic inequality that for real positive numbers a_i, the arithmetic mean is greater than or equal to the geometric mean, i.e.,

$$\frac{1}{n}\sum_{j=1}^{n} a_j \ge \left[\prod_{j=1}^{n} a_j\right]^{1/n} \tag{3.27}$$

(with equality only if $a_j = a_k \ \forall\ j,\ k$). Applying this to the sum $\sum_{j=0}^{m-1} w^j$ that appears in the factorization relation (2.1), we have, for all $w \ge 0$,

$$\sum_{j=0}^{m-1} w^j \ge m\left[\prod_{j=0}^{m-1} w^j\right]^{1/m}. \tag{3.28}$$

Now $\prod_{j=0}^{m-1} w^j = w^p$, where $p = \sum_{j=1}^{m-1} j$. Using the summation formula $\sum_{j=1}^{n} j = n(n+1)/2$, we calculate that $p = (m-1)m/2$. Hence, the inequality (3.28) for $w \ge 0$ can be written as

$$\sum_{j=0}^{m-1} w^j \ge m w^{(m-1)/2}. \tag{3.29}$$

Since $u_m = q + s(w^m - 1) = q + s(w-1)\sum_{j=1}^{m-1} w^j$, we can use the lower bound (3.29) to obtain a stronger lower bound on u_m if $w \ge 1$ (but not if $w \in [0, 1)$, since in that case the prefactor $(w-1)$ is negative). Consequently, for $w \ge 1$, substituting (3.29) into the expression for u_m, we derive the lower bound

$$u_m = q + s(w^m - 1) \ge q + ms(w-1)w^{(m-1)/2} \quad \text{for } w \ge 1. \tag{3.30}$$

Clearly, this is an improvement over the lower bound (3.1). Substituting this result into (1.4) with $u_m = u_{n(G_i')}$, we thus obtain the following improved lower bound on $Z(G, q, s, v, w)$ for $w \ge 1$ and the ferromagnetic range $v \ge 0$ (where G' is a spanning subgraph of G):

$$Z(G, q, s, v, w) \ge \sum_{G' \subseteq G} v^{e(G')} \prod_{i=1}^{k(G')} \left[q + n(G_i')s(w-1)w^{(n(G_i')-1)/2}\right] \quad \text{for } w \ge 1 \text{ and } v \ge 0. \tag{3.31}$$

Note, however, that in contrast with our previous lower bound (3.2), the right-hand side of this inequality cannot, in general, be written in terms of a zero-field Potts model partition function since the terms in the product depend explicitly on $n(G_i')$.

4 Some Thermodynamic Properties

The zero-field Potts model Hamiltonian $\mathcal{H}$ and partition function Z are invariant under the global transformation in which $\sigma_i \to g\sigma_i \ \forall i \in V$, with $g \in \mathcal{S}_q$, where $\mathcal{S}_q$ is the symmetric (= permutation) group on q objects. In the presence of the generalized external field defined in (1.2), this symmetry group of $\mathcal{H}$ and Z is reduced to the tensor product

$$\mathcal{S}_q \to \mathcal{S}_s \otimes \mathcal{S}_{q-s}. \tag{4.1}$$

This simplifies to the conventional situation in which the external field H favors or disfavors only a single spin value if $s = 1$ or $s = q - 1$, in which case the right-hand side of (4.1) is

$\mathcal{S}_{q-1}$. For s in the interval

$$2 \le s \le q-2, \tag{4.2}$$

the general model of (1.1) and (1.2) exhibits properties that are interestingly different from those of a q-state Potts model in a conventional magnetic field. For example, in the conventional case, at a given temperature T, if $H \gg |J|$, the interaction with the external field dominates over the spin-spin interaction, and if $h = \beta H$ is sufficiently large, the spins tend to be frozen to the single favored value. In contrast, here, at a given temperature T, provided that s lies in the interval (4.2), if $|H| \gg |J|$, this effectively reduces the model to (i) an s-state Potts model if $H > 0$, or (ii) a $(q-s)$-state Potts model if $H < 0$. In this limit, for given values of q and s and a given graph (say a regular lattice), there are thus, in general, four types of possible models, depending on both the sign of H and the sign of J. As an illustration of this, let us consider the case $q = 5$, $s = 2$ on (the thermodynamic limit of) a square lattice. For $H = 0$, the ferromagnetic version of the model has a first-order phase transition, with spontaneous breaking of the $\mathcal{S}_5$ symmetry, at $K_c = \ln(1+\sqrt{5}) \simeq 1.17$, while the antiferromagnetic version has no finite-temperature phase transition and is disordered even at $T = 0$ [6, 7]. For $H > 0$ and $H \gg |J|$, the theory reduces effectively to a two-state Potts model, i.e., an Ising model. Because the square lattice is bipartite, there is an elementary mapping that relates the ferromagnetic and antiferromagnetic versions of the model, and, as is well known, both have a second-order phase transition, with spontaneous symmetry breaking of the $\mathcal{S}_2 \approx \mathbb{Z}_2$ symmetry, at $|K_c| = \ln(1+\sqrt{2}) \simeq 0.881$ (where $K = \beta J$), with thermal and magnetic critical exponents $y_t = 1$, $y_h = 15/8$, described by the rational conformal field theory (RCFT) with central charge $c = 1/2$. For $H < 0$ and $|H| \gg |J|$, the theory effectively reduces to a three-state Potts model. In the ferromagnetic case, $J > 0$, this has a well-understood second-order phase transition, with spontaneous symmetry breaking of the $\mathcal{S}_3$ symmetry, at $K_c = \ln(1+\sqrt{3}) \simeq 1.01$, with thermal and critical exponents $y_t = 6/5$, $y_h = 28/15$, described by a RCFT with central charge $c = 4/5$ [6, 7, 12]. In the antiferromagnetic case, $J < 0$, the model has no finite-temperature phase transition but is critical at $T = 0$ (without frustration), with nonzero ground-state entropy per site $S/k_B = (3/2)\ln(4/3) \simeq 0.432$ [6, 13].

In particular, an interesting difference with respect to the q-state Potts model with a conventional external magnetic field appears in the case in which the spin-spin interaction is antiferromagnetic, i.e., $J < 0$. In the conventional case, there is competition between the two terms in the Hamiltonian, and resultant frustration. Here the situation is altered and depends on the chromatic number $\chi(G)$ of the graph. If $H > 0$, then there is frustration if $s < \chi(G)$, and this becomes increasingly severe as the temperature decreases, but if $s \ge \chi(G)$, then this frustration is absent, because it is possible to satisfy the antiferromagnetic short-range ordering preferred by the spin-spin interaction while also satisfying the assignments of spin values preferred by the interaction of spins with the external field. (Of course, the presence of this field does have an effect in restricting the preferred range of values of the spins.) Similarly, if $H < 0$, then there is frustration if $(q-s) < \chi(G)$ but not if $(q-s) \ge \chi(G)$. As an example, we may consider the case $q = 5$, $s = 2$ on (the thermodynamic limit of) a triangular lattice. For $H > 0$ with $H \gg |J|$, the model reduces to an Ising model, and (i) if $J > 0$, this has a symmetry-breaking second-order phase transition at $K_c = (1/2)\ln 3 \simeq 0.549$, in the same universality class as on the square lattice, while (ii) if $J < 0$, there is frustration and, as a consequence, the model has no finite-temperature phase transition, but is critical at $T = 0$, with nonzero ground-state entropy $S/k_B \simeq 0.323$ [14]. For $H < 0$ with $|H| \gg |J|$, the model reduces to a three-state Potts model, and (iii) if $J > 0$, this has a symmetry-breaking second-order phase transition at $K_c = \ln[\cos(2\pi/9) + \sqrt{3}\sin(2\pi/9)] \simeq 0.631$ [15], in the same universality class as on the square lattice; while (iv) if $J < 0$, it has a weakly first-order symmetry-breaking phase transition at $K_c \simeq -1.59$ [16, 17], with a completely ordered ground state, reflecting the fact that the chromatic number of the triangular lattice is $\chi(tri) = 3$. The more general case where $|H|$ is not $\gg |J|$ encompasses a rich variety of thermodynamic behavior depending on the signs of H and J, the ratio of $|H/J|$, the values of q and s, the dimensionality of the lattice, and, in the antiferromagnetic case, the type of d-dimensional lattice. Note that if $J = 0$, then (i) $S/k_B = \ln s$ for $H > 0$, and (ii) $S/k_B = \ln(q - s)$ for $H < 0$.

Although a one-dimensional spin system (with short-ranged spin-spin interactions, as is the case here) does not exhibit any finite-temperature phase transition, it can still serve as a worthwhile illustration of some thermodynamic properties. A simple example of this type is provided by our model on an infinite one-dimensional lattice, with either free or periodic boundary conditions. We denote a reduced, dimensionless free energy per site as $f = \lim_{n\to\infty}(1/n)\ln Z$. Then from our analysis above, we have, for the limits as $n \to \infty$ of the line and circuit graphs, $\{L\}$ and $\{C\}$,

$$f(\{L\}, q, s, v, w) = f(\{C\}, q, s, v, w) \equiv f_{1D}(q, s, v, w) = \ln(\lambda_{Z,1,0,+}), \tag{4.3}$$

where $\lambda_{Z,1,0,+}$ is given below in (5.8). From this the various thermodynamic quantities such as the internal energy, specific heat, entropy, etc. can be calculated. Also, from this, one can obtain the function $\Phi(\{G\}, q, s, w) = \lim_{n\to\infty} Ph(G_n, q, s, w)^{1/n}$ for the $n \to \infty$ limits of $G_n = L_n, C_n$. The function $\Phi(\{G\}, q, s, w)$ generalizes the ground state degeneracy per site of the zero-temperature Potts antiferromagnet, $W(\{G\}, q) = \lim_{n\to\infty} P(G, q)^{1/n}$. We have

$$\Phi(\{L\}, q, s, w) = \Phi(\{C\}, q, s, w) \equiv \Phi_{1D}(q, s, w) = (\lambda_{Z,1,0,+})|_{v=-1}. \tag{4.4}$$

We note the following reductions of Φ_{1D}, which follow from the general identities given above:

$$\Phi_{1D}(q, 0, w) = \Phi_{1D}(q, s, 1) = q - 1, \tag{4.5}$$

$$\Phi_{1D}(q, s, 0) = q - s - 1, \tag{4.6}$$

and

$$\Phi_{1D}(q, q, w) = w(q - 1). \tag{4.7}$$

With the ranges $0 \le s \le q$ and $w \ge 0$ understood, $\partial\Phi_{1D}/\partial q \ge 0$ for the nontrivial interval $q \ge 2$, reflecting the greater freedom of color assignments with increasing q. Furthermore, $\partial\Phi_{1D}/\partial w \ge 0$, as is clear from the original Hamiltonian formulation in (1.1) and (1.2). The derivative $\partial\Phi_{1D}/\partial s \ge 0$ if $w \ge 1$, and $\partial\Phi_{1D}/\partial s \le 0$ if $0 \le w \le 1$, which follows from the fact that the external field favors (disfavors) spin values in I_s if $w > 1$ ($w \in [0, 1)$). Plots of Φ_{1D} as a function of q and w for fixed s are similar to the $s = 1$ results shown in Figs. 2–4 of Ref. [3], except that the minimal value of q allowed is now s instead of 1, and the line for $w = 0$ is now $\Phi(\{L\}, q, s, 0) = q - s - 1$ rather than $q - 2$. We proceed to give exact results for $Z(G, q, s, v, w)$ and $Ph(G, q, s, w)$ for several families of graphs.

5 Path Graph L_n

The path graph L_n is the graph consisting of n vertices with each vertex connected to the next one by one edge. One may picture this graph as forming a line, and in the physics literature this is commonly called a line graph. We use the alternate term "path graph" here because in mathematical graph theory the line graph $L(G)$ of a graph G refers to a different object (namely the graph obtained by an ismorphism in which one maps the edges of G to the vertices of $L(G)$ and connects these resultant vertices by edges if the edges of G are connected to the same vertex of G). For $n \geq 2$, the chromatic number is $\chi(L_n) = 2$. In [4] we gave some illustrative calculations of $Z(L_n, q, s, v, w)$. Here we present a general formula for this partition function. Let

$$T_{Z,1,0} = \begin{pmatrix} q - s + v & sw \\ q - s & w(s + v) \end{pmatrix} \tag{5.1}$$

$$H_{1,0} = \begin{pmatrix} 1 & 0 \\ 0 & sw \end{pmatrix} \tag{5.2}$$

$$\omega_1 = \begin{pmatrix} q - s \\ 1 \end{pmatrix} \tag{5.3}$$

and

$$s_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{5.4}$$

Then

$$Z(L_n, q, s, v, w) = \omega_1^T H_{1,0} (T_{Z,1,0})^{n-1} s_1 \tag{5.5}$$

and $Ph(L_n, q, s, w) = Z(L_n, q, s, -1, w)$. It is straightforward to verify that our result for $Z(L_n, q, s, v, w)$ satisfies the relation (2.30). We note that

$$\det(T_{Z,1,0}) = v(q + v)w, \tag{5.6}$$

independent of s, and

$$\mathrm{Tr}(T_{Z,1,0}) = q - s + v + w(s + v). \tag{5.7}$$

The eigenvalues of $T_{Z,1,0}$ are the same as the eigenvalues with coefficients of degree $d = 0$ for the circuit graph C_n given in (5.3) of Ref. [4], namely

$$\lambda_{Z,1,0,\pm} = \frac{1}{2}\left[q - s + v + w(s + v) \pm \left[\{q - s + v + w(s + v)\}^2 - 4vw(q + v) \right]^{1/2} \right]. \tag{5.8}$$

Thus, we can also write

$$Z(C_n, q, s, v, w) = \mathrm{Tr}[(T_{Z,1,0})^n] + (s - 1)(vw)^n + (q - s - 1)v^n. \tag{5.9}$$

The graphs L_n, C_n, and, more generally, lattice strip graphs of some transverse width L_y and length $L_x = m$ are examples of recursive families of graphs, i.e., graphs G_m that have the property that G_{m+1} can be constructed by starting with G_m and adding a given graph H or, if necessary, cutting and gluing in H. For these graphs, $Z(G_m, q, s, v, w)$ has the structure of a sum of coefficients that are independent of the length m multiplied by m'th powers of some algebraic functions. The results for transfer matrices for the case $s = 1$ in Ref. [2] elucidated this structure for $s = 1$, and our calculation of $Z(C_n, q, s, v, w)$ in Ref. [4] and $Z(L_n, q, s, v, w)$ here elucidate this structure for general s. Note that, by (2.9), the term in $Z(C_n, q, s, v, w)$ of highest order in v is $v^n u_n = [q + s(w-1)\sum_{j=0}^{n-1} w^j]v^n$, part of which gives rise to the last two terms in (5.9). We note that for $s = 0$ or $w = 1$, one can check that our expressions for $Z(L_n, q, s, v, w)$ and $Z(C_n, q, s, v, w)$ simplify, respectively, to $Z(L_n, q, v) = q(q+v)^{n-1}$ and $Z(C_n, q, v) = (q+v)^n + (q-1)v^n$. Going from the case of $sw(w-1) = 0$ to $sw(w-1) \neq 0$, $Z(L_n, q, s, v, w)$ expands from a sum of one power to a sum involving two powers, and $Z(C_n, q, s, v, w)$ expands from a sum of two powers to a sum of four powers.

As our exact solutions for $Z(L_n, q, s, v, w)$ and $Z(C_n, q, s, v, w)$ show, the field-dependent Potts partition functions $Z(G, q, s, v, w)$ do not, in general, have any common factor. This contrasts with the case of the zero-field Potts partition function, which always has an overall factor of q. Similarly, in the $v = -1$ special case defining the set-weighted chromatic polynomial, the resultant polynomials $Ph(G, q, s, w)$ do not, in general, have a common factor. For special values of s, $Ph(G, q, s, w)$ may reduce to a form with a common factor. The case $s = 0$ (and the case $w = 1$) for which this reduces to the conventional chromatic polynomial is well-known; in this case $P(G, q)$ has, as a common factor, $\prod_{j=0}^{\chi(G)-1}(q-j)$. Similarly, for $s = q$, $Ph(G, q, q, w)$ has this common factor multipled by w^n. For the special case $s = 1$ and for a connected graph G with at least one edge, it was shown in Ref. [3] that $Ph(G, q, 1, w)$ contains a factor $(q-1)$. However, it is not true that for a special case such as $s = 2$, a connected graph G with at least one edge contains a factor of $(q-s)$. For example, using the elementary result

$$Z(L_2, q, s, v, w) = s(s+v)w^2 + 2s(q-s)w + (q-s)(q-s+v), \tag{5.10}$$

one sees that $Ph(L_2, q, 1, w) = (q-1)(q-2+2w)$, but $Ph(L_2, q, 2, w) = 2w^2 + 4(q-2)w + (q-2)(q-3)$, which has no common factor.

6 Complete Graphs K_n

The complete graph K_n is the graph with n vertices such that each vertex is connected to every other vertex by one edge. The chromatic number is $\chi(K_n) = n$ and the number of edges is $e(K_n) = \binom{n}{2}$. The (conventional, unweighted) chromatic polynomial is

$$P(K_n, q) = \prod_{j=0}^{n-1}(q-j). \tag{6.1}$$

For our later calculations, we will need our previous result for $Ph(K_n, q, s, w)$ from Ref. [4], which we mention here. We introduce a symbol $x_\theta \equiv x\theta(x)$, where $\theta(x)$ is the step function from $\mathbb{R} \to \{0, 1\}$ defined as $\theta(x) = 1$ if $x > 0$ and $\theta(x) = 0$ if $x \le 0$. Our result is [4]

$$Ph(K_n, q, s, w) = \sum_{\ell=0}^{n} \beta_{K_n,\ell}(q, s)w^\ell \tag{6.2}$$

where

$$\beta_{K_n,\ell}(q, s) = \binom{n}{\ell}\left[\prod_{i=0}^{(\ell-1)_\theta}(s-i)\right]\left[\prod_{j=0}^{(n-\ell-1)_\theta}(q-s-j)\right]. \tag{6.3}$$

Here it is understood that if the upper index on either of the two products in (6.3) is negative, that product is absent, so that the first product is absent for $\ell = 0$ and the second one is absent for $\ell = n$. Note that

$$\beta_{K_n,\ell}(q,s) = \beta_{K_n,n-\ell}(q,q-s), \tag{6.4}$$

in agreement with the general symmetry (2.18). Substituting this in (6.2) shows explicitly that our result for $Ph(K_n,q,s,w)$ satisfies the symmetry relation (2.13). Note that K_n is not a recursive family of graphs, so one does not expect $Ph(K_n,q,s,w)$ to have the form of a sum of coefficients multiplied by powers of certain algebraic functions, and it does not, in contrast to $Ph(G_n,q,s,w)$ for recursive families G_n such as C_n or L_n.

The calculation of $Ph(K_n,q,s,w)$ for the cases K_1 and $K_2 = L_2$ are elementary. For $K_3 = C_3$ our general formula (6.2) yields

$$Ph(K_3,q,s,w) = P(K_3,s)w^3 + 3s(s-1)(q-s)w^2 + 3s(q-s)(q-s-1)w + P(K_3,q-s) \tag{6.5}$$

while for K_4 we have

$$\begin{aligned} Ph(K_4,q,s,w) = {} & P(K_4,s)w^4 + 4s(s-1)(s-2)(q-s)w^3 \\ & + 6s(s-1)(q-s)(q-s-1)w^2 \\ & + 4s(q-s)(q-s-1)(q-s-2)w + P(K_4,q-s). \end{aligned} \tag{6.6}$$

7 p-Wheel Graphs $Wh^{(p)} = K_p + C_{n-p}$

The p-wheel graph $Wh_n^{(p)}$ is defined as

$$Wh_n^{(p)} = K_p + C_{n-p}, \tag{7.1}$$

i.e., the join of the complete graph K_p with the circuit graph C_{n-p}. (Given two graphs G and H, the join, denoted $G+H$, is defined as the graph obtained by joining each of the vertices of G to each of the vertices of H). (Here and below, no confusion should result from the use of the symbol H for a graph and H for the external field; the meaning will be clear from context.) The family of $Wh_n^{(p)}$ graphs is a recursive family. For $p=1$, $Wh_n^{(1)}$ is the wheel graph. The central vertex can be regarded as forming the axle of the wheel, while the $n-1$ vertices of the C_{n-1} and their edges form the outer rim of the wheel. This is well-defined for $n \ge 3$, and in this range the chromatic number is $\chi(Wh_n) = 3$ if n is odd and $\chi(Wh_n) = 4$ if n is even. Although K_p is not defined for $p=0$, we may formally define $Wh_n^{(0)} \equiv C_n$. For the zero-field case, i.e., for the usual, unweighted chromatic polynomial and for an arbitrary graph G,

$$P(K_p+G,q) = P(K_p,q)P(G,q-p) = q_{(p)}P(G,q-p), \tag{7.2}$$

where $q_{(m)}$ is the falling factorial, defined as

$$q_{(m)} = \prod_{j=0}^{m-1}(q-j). \tag{7.3}$$

This result is a consequence of the fact that in assigning colors to the p vertices of K_p, one must use p different colors, and then, because of the join condition, one must select from the other $q-p$ colors to color the vertices of G. In particular, for $Wh^{(p)}$, this gives

$$P(Wh_n^{(p)},q) = P(K_p,q)P(C_{n-p},q-p) = q_{(p)}\Big[(q-1-p)^{n-p} + (q-1-p)(-1)^{n-p}\Big]. \tag{7.4}$$

Note that, for arbitrary p, this chromatic polynomial consists of the prefactor times the sum of the $(n-p)$'th powers of $N_{Wh^{(p)},\lambda} = 2$ terms. For $p=1$, this number can be seen to be the $L_y = 1$ special case of a general formula in (3.2.15) of Ref. [18] for the join of K_1 with a width-L_y cyclic strip.

For the weighted-set chromatic polynomial, we generalize this coloring method as follows. Consider first $K_1 + G$. There are two possible types of choices for the color to be assigned to the vertex of K_1. One type is to choose this color to lie in the set I_s. There are s ways to make this choice, and each gets a weighting factor of w. For each choice, one then performs the proper coloring of the vertices of G with the remaining $q-1$ colors, of which only $s-1$ can be used from the set I_s; this is determined by $Ph(G,q-1,s-1,w)$. The second type of coloring is to choose the color assigned to the K_1 vertex to lie in the orthogonal set $I_s^{\perp}$. There are $(q-s)$ ways to make this choice, and since this is not the weighted set, there is no weighting factor of w. For each such choice, one then performs the proper coloring of the vertices of G with the remaining $q-1$ colors, of which all s colors in the set I_s are available, but only $q-s-1$ colors in the orthogonal set $I_s^{\perp}$ are available. This yields the result

$$Ph(K_1+G,q,s,w) = swPh(G,q-1,s-1,w) + (q-s)Ph(G,q-1,s,w). \tag{7.5}$$

To calculate $Ph(K_p+G,q,s,w)$ for a given graph G, one first carries out the proper coloring of K_1+G, using the result (7.5). One then joins the next vertex of K_p to K_1+G to get K_2+G, using the relation $K_1 + (K_r+G) = K_{r+1}+G$ and iteratively applies (7.5). One continues in this manner to carry out the proper coloring of the full join K_p+G. This yields

$$Ph(K_p+G,q,s,w) = \sum_{\ell=0}^{p} \beta_{K_p,\ell}(q,s)Ph(G,q-p,s-\ell,w)w^{\ell}. \tag{7.6}$$

Utilizing this coloring method, we have calculated $Ph(Wh_n^{(p)},q,s,w)$ for arbitrary n. Let us define

$$a(p,q,s,w) = q-s-(p+1)+(s-1)w = q-(p+1)+s(w-1)-w \tag{7.7}$$

and

$$\lambda_{Wh^{(p)},\ell,\pm}(q,s,w) = \frac{1}{2}\Big[a(p,q,s-\ell,w) \pm [a(p,q,s-\ell,w)^2 + 4w(q-p-1)]^{1/2}\Big] \quad \text{for } 0 \le \ell \le p. \tag{7.8}$$

We note that for $\ell = 0$, these $\lambda_{Wh^{(p)},\ell,\pm}(q,s,w)$ are equal to the $v=-1$ special case of $\lambda_{Z,1,0,j}$ given in (5.3) of our earlier Ref. [4] for the circuit graph with the replacement of q by $q-p$ (and with $j=1,2$ corresponding to $\pm$ here). This is in accord with the fact that the effect of the join of K_p with G is that the proper q-coloring of G can only use $q-p$ of the original q colors. We define two additional terms that do not depend on q or s,

$$\lambda_{Wh^{(p)},2p+3} = -w \tag{7.9}$$

and

$$\lambda_{Wh^{(p)},2p+4} = -1. \tag{7.10}$$

The total number of λ's for $Ph(Wh_n^{(p)}, q, s, w)$ is thus

$$N_{Ph(Wh^{(p)}),\lambda} = 2(p+2). \tag{7.11}$$

Note that in contrast to the unweighted chromatic polynomial of $Wh_n^{(p)}$, where the number of λ's, $N_{P(Wh^{(p)}),\lambda} = 2$, is independent of p, here this number depends on p. In terms of these quantities, we find, for the weighted-set chromatic polynomial for $Wh_n^{(p)}$, the result

$$\begin{aligned} Ph(Wh_n^{(p)}, q, s, w) &= \sum_{\ell=0}^{p} \beta_{K_p,\ell}(q,s) \Bigg[[\lambda_{Wh^{(p)},\ell,+}(q,s,w)]^{n-p} \\ &\quad + [\lambda_{Wh^{(p)},\ell,-}(q,s,w)]^{n-p} \Bigg] w^{\ell} \\ &\quad + \left[\sum_{\ell=0}^{p} \beta_{K_p,\ell}(q,s)(s-\ell-1) w^{\ell} \right] (-w)^{n-p} \\ &\quad + \left[\sum_{\ell=0}^{p} \beta_{K_p,\ell}(q,s)(q-s-p+\ell-1) w^{\ell} \right] (-1)^{n-p}. \end{aligned} \tag{7.12}$$

This formula applies for integer $p \ge 1$ and also for $p = 0$ if one sets $\beta_{K_p,\ell}(q,s) \equiv \delta_{\ell,0}$ for $p = 0$. It can be checked that for $p = 0$, (7.12) reduces to our result for $Ph(C_n, q, s, w)$ given as the special $v = -1$ case of (5.3)–(5.5) in Ref. [4]. It can also be verified that for $p = 1$ and $s = 1$, (7.12) reduces to the result given for this case in (3.30)–(3.32) in Ref. [3]. Furthermore, since the graph $Wh_4^{(1)} = K_1 + K_3 = K_4$, it follows that $Ph(Wh_4^{(1)}, q, s, w) = Ph(K_4, q, s, w)$. The symmetry (2.13) is realized as follows: the summation on the first line of (7.12) goes into itself, while the sum of the expressions on the two subsequent lines of (7.12) transforms into itself with the replacement of w by w^{-1} in these expressions and the prefactor w^n appearing overall. One could also study $Z(Wh_n^{(p)}, q, s, v, w)$, but we have focused here on $Ph(Wh^{(p)}, q, s, w)$, since its calculation can be performed by combinatoric methods associated with the proper q-coloring condition. We give some explicit examples of set-weighted chromatic polynomials $Ph(Wh_n^{(p)}, q, s, w)$ obtained from our general formula (7.12) in the first appendix.

Following our notation in Ref. [4] and earlier works, the $n \to \infty$ limit of a family of n-vertex graphs G_n is denoted $\{G\}$ and the continuous accumulation set of the zeros of $Ph(G_n, q, s, w)$ in the complex q plane is denoted $\mathcal{B}_q$. For recursive families of graphs, this locus is determined as the solution of the equality in magnitude of two (or more) λ's of dominant magnitude, as a function of q (with other variables held fixed). The other loci $\mathcal{B}_v$, etc. are defined in an analogous manner. These loci are typically comprised of curves and possible line segments. For studies of the $n \to \infty$ limit of chromatic polynomials and their generalization to weighted-set chromatic polynomials, the locus $\mathcal{B}_q$ is of primary interest. Depending on the family of graphs, the locus $\mathcal{B}_q$ may or may not cross the real q axis. If it does cross the real q axis, we denote the maximum (finite) point at which it crosses this axis as q_c. Extending our previous result for the $p = 0$ case of $\{G\} = \{Wh^{(p)}\}$ in (7.17) of Ref.

[4], we find the following result for general p:

$$q_c = 2 + p + \frac{s(1-w)}{1+w} \quad \text{for } \{G\} = \{Wh^{(p)}\} \text{ and } 0 \le w \le 1 \text{ and } 1 \le s \le p+2. \quad (7.13)$$

Regarding connections of this general formula to previously determined special cases, (i) for $s = 0$ or $w = 1$, this reduces to the result $q_c = 2 + p$ for the $n \to \infty$ limit of the chromatic polynomial $P(Wh^{(p)}, q)$ given in (22) of Ref. [19]; (ii) for $p = 0$, this reduces to the result for the $n \to \infty$ of $Ph(C_n, q, s, w)$ given in (7.17) of Ref. [4], and (iii) for $s = 1$, this reduces to the result for the $n \to \infty$ limit of $Ph(Wh^{(1)}, q, 1, w)$ given in (10.1) of Ref. [3] (with the obvious notation change $\{C\} \to \{Wh\}$). For the relevant interval $0 \le w \le 1$, the value of q_c in (7.13) is (a) greater than the value $q_c = 2 + p$ for the unweighted chromatic polynomial; (b) a monotonically increasing function of s for fixed w in this DFSCP interval; and (c) a monotonically decreasing function of w. These properties are consequences of the greater suppression of color values in the set I_s as w decreases in the DFSCP interval, finally restricting the vertex coloring to use colors from the orthogonal set $I_s^{\perp}$ as w reaches 0. Thus, as w decreases from 1 to 0, q_c increases continuously from $2 + p$ to $2 + p + s$. In contrast, the left-hand part of the boundary locus $\mathcal{B}_q$ changes discontinuously; as w decreases by an arbitrarily small amount below 1, the point on the left where $\mathcal{B}_q$ crosses the real q axis jumps discontinuously from $q = p$ to $q = p + s$. This behavior is in agreement with the fact that in the two limits $w = 1$ and $w = 0$, $\mathcal{B}_q$ is comprised, respectively, of the unit circle centered at $q = 1 + p$ and the unit circle centered at $q = 1 + s + p$. The change in the nature of the locus for $s > 2 + p$ follows via the corresponding generalization of the analysis in Ref. [4] to $p \ge 0$.

8 Effect of Multiple Edges in a Graph

Consider a loopless graph $G = (V, E)$. Replace each edge with ℓ edges joining the same pair of vertices and denote the resultant graph as $G_{\ell e}$. Then the following is a theorem:

$$Z(G_{\ell e}, q, s, v, w) = Z(G, q, s, v_\ell, w), \quad \text{where } v_\ell = (v+1)^\ell - 1. \quad (8.1)$$

Clearly, if $v = 0$, then $Z(G, q, s, v, w) = (q - s + sw)^n$, independent of the edge set E of G. Hence, in this case, the operation of replacing each edge by ℓ copies of the edge has no effect on the partition function. This is seen at an analytic level via the property that if $v = 0$, then also $v_\ell = 0$ for any (positive integer) ℓ. Further, for $v = -1$, where $Z(G, q, s, v, w)$ reduces to the weighted-set chromatic polynomial $Ph(G, q, s, w)$, the proper q-coloring constraint is the same regardless of whether a given edge is replicated or not, so again the replication does not affect this polynomial. In (8.1), this follows because if $v = -1$, then also $v_\ell = -1$ for any (positive integer) ℓ. Combining these results, we note that

$$v_\ell - v \text{ contains the factor } v(v+1). \quad (8.2)$$

Consequently, for any graph G with at least one edge (so that the operation of edge replication is not vacuous) and for positive integer ℓ,

$$Z(G_{\ell e}, q, s, v, w) - Z(G, q, s, v_\ell, w) \text{ contains the factor } v(v+1). \quad (8.3)$$

9 Effects of Deletion and Contraction of Edges

As noted above in Sect. 2, in general, neither $Z(G, q, s, v, w)$ nor $Ph(G, q, s, w)$ satisfies the respective deletion-contraction relation. It is of interest to investigate how these polynomials deviate from the deletion-contraction relation. A natural measure of this deviation for a graph G is [4]

$$[\Delta Z(G, e, q, s, v, w)]_{DCR} = Z(G, q, s, v, w) - \Big[Z(G-e, q, s, v, w) + vZ(G/e, q, s, v, w)\Big]. \quad (9.1)$$

We also define $[\Delta Ph(G, e, q, s, w)]_{DCR} \equiv [\Delta Z(G, e, q, s, -1, w)]_{DCR}$. We showed that [4]

$$[\Delta Z(G, e, q, s, v, w)]_{DCR} \text{ contains the factor } svw(w-1), \quad (9.2)$$

and hence $[\Delta Ph(G, e, q, s, w)]_{DCR}$ contains a factor of $sw(w-1)$. A particularly elegant general formula can be obtained for this deviation in the case of the family of star graphs, S_n, i.e., graphs consisting of one central vertex with $n-1$ other vertices, each of which is only connected to this central vertex. We find (for the nontrivial range $n \ge 2$)

$$[\Delta Z(S_n, q, s, v, w)]_{DCR} = svw(w-1)[q + s(w-1) + wv]^{n-2}. \quad (9.3)$$

10 Cycle Measure

In view of the relation (2.30), one can define the following function, which serves as a measure of the presence of cycles in a graph G:

$$[\Delta Z(G, q, s, v, w)]_{cycles} = Z(G, q, s, v, w) - s^n Z(G, q', 1, v', w), \quad (10.1)$$

where q' and v' were defined in (2.29). Clearly, this difference vanishes if $s = 1$, so, since it is a rational function,

$$[\Delta Z(G, q, s, v, w)]_{cycles} \text{ contains the factor } (s-1). \quad (10.2)$$

In Ref. [4] we derived the result

$$[\Delta Z(C_n, q, s, v, w)]_{cycles} = \frac{(s-1)u_n v^n}{s} = \frac{(s-1)(q-s+sw^n)v^n}{s}. \quad (10.3)$$

This reflects the fact that C_n contains one cycle.

Here we present another example of this difference function. Let us define a path graph with n vertices and each edge replaced by ℓ edges joining the same adjacent vertices as $L_{n,\ell}$. Note that $L_{2,2} = C_2$. For $L_{3,2}$ we calculate

$$\begin{aligned}
&[\Delta Z(L_{3,2}, q, s, v, w)]_{cycles} \\
&\quad = \frac{(s-1)v^2}{s}\Big[s(2s^2 + v^2 s + 4vs + v^2)w^3 + 2s^2(q-s)w^2 + 2s^2(q-s)w \\
&\qquad + (q-s)(-2s^2 + v^2 s + 4vs + 2sq + v^2)\Big]. \quad (10.4)
\end{aligned}$$

11 Use of $Z(G,q,s,v,w)$ and $Ph(G,q,s,w)$ to Distinguish Between Tutte-Equivalent and Chromatically Equivalent Graphs

11.1 General

Two graphs G and H are defined to be (i) chromatically equivalent if they have the same chromatic polynomial, and (ii) Tutte-equivalent if they have the same Tutte polynomial, or equivalently, zero-field Potts model partition function. Here the Tutte polynomial $T(G,x,y)$ of a graph G is defined as

$$T(G,x,y) = \sum_{G' \subseteq G} (x-1)^{k(G')-k(G)}(y-1)^{c(G')}, \tag{11.1}$$

where G' is a spanning subgraph of G (and $c(G')$ and $k(G')$ were defined above as, respectively, the number of linearly independent cycles and the number of connected components of G'). This is equivalent to the zero-field Potts model partition function, via the relation

$$Z(G,q,v) = (x-1)^{k(G)}(y-1)^n T(G,x,y), \tag{11.2}$$

where $y = v+1$ as in (1.3) and $x = 1+(q/v)$. The Tutte polynomial is of considerable interest in mathematical graph theory, since it encodes much information about a graph. However, although it distinguishes between many graphs, there exist other pairs of graphs G and H that are different but have the same Tutte polynomial. An important property of our generalized field-dependent Potts model partition function $Z(G,q,s,v,w)$ is that it can distinguish between many Tutte-equivalent graphs. Similarly, an important property of the weighted-set chromatic polynomial is that it can distinguish between many chromatically equivalent graphs. We study this further in this section. This property is true for all w and s values except the special values $w=1$, $w=0$, $s=0$, and $s=q$, for which $Z(G,q,s,v,w)$ is reducible to a zero-field Potts partition function (as well as the trivial case $v=0$) and similarly for $Ph(G,q,s,w)$. reducible to a chromatic polynomial. In Ref. [4] we proved that for any two Tutte-equivalent graphs G and H,

$$Z(G,q,s,v,w) - Z(H,q,s,v,w) \text{ contains the factor } s(q-s)vw(w-1). \tag{11.3}$$

In the following, we will generally phrase our analysis in terms of how the field-dependent Potts partition function distinguishes between Tutte-equivalent graphs; the special cases of the various expressions for $v=-1$ then show how the weighted-set chromatic polynomial distinguishes between different chromatically equivalent graphs.

11.2 Tree Graphs

A class of Tutte-equivalent (and, hence also chromatically equivalent) graphs of particular interest is comprised of tree graphs, generically denoted T_n. For these, $T(T_n,x,y) = x^{n-1}$, so

$$Z(T_n,q,v) = q(q+v)^{n-1} \quad \text{and} \quad P(T_n,q) = q(q-1)^{n-1}. \tag{11.4}$$

Note that $e(T_n) = n-1$ (and a tree graph cannot have any multiple edges). There is only one tree graph with $n=1$ vertex, one with $n=2$ vertices, and one with $n=3$ vertices. There are two different tree graphs with $n=4$ vertices, namely the path graph, L_4, and the star graph, S_4. Enumerations of tree graphs with larger numbers of vertices are given, e.g., in

Refs. [20, 21]. Let us consider two different n-vertex tree graphs (which thus have $n \ge 4$), denoted G_t and H_t. Since these have the same number of edges, inspection of the general equation (2.9) shows that for the difference $Z(G_t, q, s, v, w) - Z(H_t, q, s, v, w)$, not only the v^0 and v^n terms, but also the v^1 terms cancel. Hence,

$$Z(G_t, q, s, v, w) - Z(H_t, q, s, v, w) \text{ contains the factor } v^2. \tag{11.5}$$

We recall that $I_s \subseteq I_q$, so that $0 \le s \le q$, and that $w \ge 0$, as follows for any physical field H. These properties will be understood implicitly in the following. As preparation for the derivation of an inequality concerning $Z(G_t, q, s, v, w)$ for S_n and L_n graphs, it is useful to give some explicit examples. Let us consider the two tree graphs with $n = 4$ vertices, namely S_4 and L_4. In the following, we will usually omit the arguments q, s, v, w in $Z(G, q, s, v, w)$ for brevity of notation. We have given exact expressions for $Z(S_n)$ in (3.5) of Ref. [4] and for $Z(L_n)$ in (5.5) above. For our present purposes, we focus on the expressions in terms of the spanning subgraph expansion. For S_4, this is

$$Z(S_4) = u_1^4 + 3vu_2u_1^2 + 3v^2u_3u_1 + v^3u_4, \tag{11.6}$$

while for L_4 we have

$$Z(L_4) = u_1^4 + 3vu_2u_1^2 + v^2(2u_3u_1 + u_2^2) + v^3u_4. \tag{11.7}$$

The difference in the structure of the term proportional to v^2 arises from the differences in the spanning subgraphs with two edges in S_4 and L_4. Hence,

$$Z(S_4) - Z(L_4) = v^2(u_3u_1 - u_2^2) = v^2 s(q-s)w(w-1)^2. \tag{11.8}$$

Since the last expression will appear as a factor in the differences $Z(G_t) - Z(H_t)$ to be presented below, we give it a symbol:

$$\mu \equiv s(q-s)v^2w(w-1)^2 \tag{11.9}$$

and note that

$$\mu \ge 0, \tag{11.10}$$

so that $Z(S_4) - Z(L_4) \ge 0$.

There are three different tree graphs with $n = 5$ vertices: S_5, L_5, and a graph that we denote as Y_5, which has the form of a Y, with the vertical part made up of three vertices and two edges (shown in Fig. 1 of Ref. [3]). The graph Y_n is the generalization of this graph in which the vertical part is comprised of $n-2$ vertices forming a path graph P_{n-2} (so that $Y_4 = S_4$). The spanning subgraph expansions for these graphs, in order of decreasing maximal vertex degree, are

$$Z(S_5) = u_1^5 + 4vu_2u_1^3 + 6v^2u_3u_1^2 + 4v^3u_4u_1 + v^4u_5, \tag{11.11}$$

$$Z(Y_5) = u_1^5 + 4vu_2u_1^3 + 2v^2(2u_3u_1^2 + u_2^2u_1) + v^3(3u_4u_1 + u_3u_2) + v^4u_5, \tag{11.12}$$

and

$$Z(L_5) = u_1^5 + 4vu_2u_1^3 + 3v^2(u_3u_1^2 + u_2^2u_1) + 2v^3(u_4u_1 + u_3u_2) + v^4u_5. \tag{11.13}$$

Thus, for the differences, we have

$$\begin{aligned} Z(S_5) - Z(Y_5) &= 2v^2(u_3u_1^2 - u_2^2u_1) + v^3(u_4u_1 - u_3u_2) \\ &= \mu[2u_1 + v(w+1)], \end{aligned} \tag{11.14}$$

$$\begin{aligned} Z(S_5) - Z(L_5) &= 3v^2(u_3u_1^2 - u_2^2u_1) + 2v^3(u_4u_1 - u_3u_2) \\ &= \mu[3u_1 + 2v(w+1)], \end{aligned} \tag{11.15}$$

and

$$\begin{aligned} Z(Y_5) - Z(L_5) &= v^2(u_3u_1^2 - u_2^2u_1) + v^3(u_4u_1 - u_3u_2) \\ &= \mu[u_1 + v(w+1)]. \end{aligned} \tag{11.16}$$

Now (remembering that $0 \le s \le q$ and $w \ge 0$), for the ferromagnetic range $v \ge 0$, for non-negative a and b, one has

$$au_1 + bv(w+1) \ge 0. \tag{11.17}$$

Hence, for the ferromagnetic case, each of the differences $Z(S_5) - Z(Y_5)$, $Z(S_5) - Z(L_5)$, and $Z(Y_5) - Z(L_5)$ is non-negative.

From these explicit examples, one sees that the origin of these inequalities can be traced to inequalities among products of the u_r's. We proceed to prove two lemmas and then a general theorem. Our first lemma is

$$u_{n-1}u_1 \ge u_{n-\ell}u_\ell \quad \text{for } n \ge 2 \text{ and } 2 \le \ell \le n-2. \tag{11.18}$$

To verify this lemma, we expand and factor the given expression:

$$\begin{aligned} u_{n-1}u_1 - u_{n-\ell}u_\ell &= s(q-s)w(1 + w^{n-2} - w^{\ell-1} - w^{n-\ell-1}) \\ &= s(q-s)w(w^{n-\ell-1} - 1)(w^{\ell-1} - 1) \\ &= s(q-s)w(w-1)^2 \left[\sum_{i=0}^{n-\ell-2} w^i\right]\left[\sum_{j=0}^{\ell-2} w^j\right] \ge 0. \end{aligned} \tag{11.19}$$

This lemma shows that the difference $u_3u_1 - u_2^2$ that appears multiplying v^2 in (11.8), (11.14), (11.15), and (11.16) is nonnegative, and similarly that the difference $u_4u_1 - u_3u_2$ that appears multiplying v^3 in the last three of these equations is nonnegative.

Differences of the form $Z(G_t) - Z(H_t)$ for higher values of n involve differences of higher products of u_r factors, and there is an analogous inequality for these products. We prove this as a second lemma. Let us consider a generic term in (1.4), for the spanning subgraph $G' = \bigoplus G_i'$ with $k(G')$ connected components, G_i', each with $n(G_i')$ vertices. This has the form (2.4) satisfying the relation (2.5). Our second lemma is, with $\ell = n - k(G') + 1$,

$$u_\ell u_1^{n-\ell} \ge \prod_{j=1}^{k(G')} u_{n(G_j')} \quad \text{for } n \ge 2 \text{ and } 1 \le \ell \le n, \text{ i.e., } 1 \le k(G') \le n. \tag{11.20}$$

For example, for the case $n = 6$, this lemma yields the inequalities $u_4u_1^2 \ge u_3^2$, $u_4u_1^2 \ge u_2^3$, and $u_4u_1^2 \ge u_4u_2$. This lemma is proved by the same method as Lemma 1.

Combining the expression for $Z(S_n, q, s, v, w)$ in the first line of (3.21) with our other results above, we have the following theorem: For the ferromagnetic case,

$$Z(S_n, q, s, v, w) - Z(T_n, q, s, v, w) \ge 0 \quad \text{for } v \ge 0 \tag{11.21}$$

for any tree graph T_n. This is proved by applying the two lemmas above to the terms in the spanning subgraph expansions of these partition functions for S_n and a generic tree graph T_n. In the second appendix we give further explicit results for differences of field-dependent partition functions for tree graphs with $n = 6$ vertices.

The difference $Z(Y_5) - Z(L_5)$ in (11.16) (where we omit the arguments for brevity of notation) can also be understood using the recursive relation for $n \ge 5$:

$$\begin{aligned} Z(Y_n) - Z(L_n) &= \sum_{j=1}^{n-4} v^{j-1} u_j [Z(Y_{n-j}) - Z(L_{n-j})] \\ &\quad + v^{n-4} \left(\sum_{j=0}^{n-4} w^j \right) [Z(Y_4) - Z(L_4)], \end{aligned} \tag{11.22}$$

where $Z(Y_4) - Z(L_4) = Z(S_4) - Z(L_4) = \mu$ was given in (11.8). For the ferromagnetic range $v \ge 0$, each term on the right-hand side of (11.22) is nonnegative, and hence this proves the inequality

$$Z(Y_n, q, s, v, w) - Z(L_n, q, s, v, w) \ge 0 \quad \text{for } v \ge 0. \tag{11.23}$$

Combining (11.21) and (11.23), we have

$$Z(S_n, q, s, v, w) \ge Z(Y_n, q, s, v, w) \ge Z(L_n, q, s, v, w) \quad \text{for } v \ge 0. \tag{11.24}$$

11.3 Properties of Graphs Intersecting in a Complete Graph

One class of chromatically equivalent graphs consists of graphs whose chromatic polynomials can be shown to be equal by an application of the complete graph intersection theorem. We recall this theorem. Let us consider a graph G that has the property of being composed of the union of two subgraphs, $G = G_1 \cup G_2$, such that $G_1 \cap G_2 = K_m$ for some m. In the rest of this subsection, we assume that G has this property. Then $P(G, q)$ satisfies the relation

$$P(G, q) = \frac{P(G_1, q) P(G_2, q)}{P(K_m, q)}. \tag{11.25}$$

This is sometimes called the complete-graph intersection theorem (KIT) for chromatic polynomials. In contrast, in general, $Ph(G, q, s, w)$ is not equal to $Ph(G_1, q, s, w)$ $Ph(G_2, q, s, w)/Ph(K_m, q, s, w)$. This equality holds only for the four values $w = 1$, $w = 0$, $s = 0$, and $s = q$ where $Ph(G, q, s, w)$ reduces to a chromatic polynomial. As a measure of the deviation from equality, we define

$$[\Delta Ph(G, q, s, w)]_{KIT} \equiv Ph(G, q, s, w) - \frac{Ph(G_1, q, s, w) Ph(G_2, q, s, w)}{Ph(K_m, q, s, w)}. \tag{11.26}$$

Let us consider a graph with $n = 5$ vertices, denoted G_{LKL}, comprised of a triangle $K \equiv K_3$ with two line segments L, each of length one edge, emanating outward from two vertices of the triangle. This graph LKL has $n=5$, $e=5$ and $c=1$. A second graph, G_{KLL}, also with $n=5$, $e=5$ and $c=1$, is comprised of a triangle K_3 with a line segment two edges long emanating outward from one vertex of the triangle. These graphs are Tutte-equivalent, with

$$T(G_{LKL},x,y)=T(G_{KLL},x,y)=x^2(x+x^2+y), \tag{11.27}$$

or equivalently,

$$Z(G_{LKL},q,v)=Z(G_{KLL},q,v)=q(q+v)(q^2+3qv+3v^2+v^3). \tag{11.28}$$

It follows that these graphs are also chromatically equivalent, with chromatic polynomial

$$P(G_{LKL},q)=P(G_{KLL},q)=q(q-1)^3(q-2). \tag{11.29}$$

In contrast, the field-dependent Potts partition function and the weighted-set chromatic polynomial successfully distinguish between these graphs. For the LKL graph we calculate

$$\begin{aligned}
Z(G_{LKL},q,s,v,w) &= Z(G_{LKL},s,v)w^5+s(q-s)(s+v)(5s^2+10sv+7v^2+2v^3)w^4\\
&\quad +s(q-s)\Big[(q-s+2)v^3+(7q-s)v^2+10s^2(q-s-v)\\
&\quad +15sqv\Big]w^3+s(q-s)\Big[(s+2)v^3+(6q+s)v^2\\
&\quad +10(q-s)^2(s-v)+15(q-s)qv\Big]w^2+s(q-s)(q-s+v)\\
&\quad \times\Big[5(q-s)^2+10(q-s)v+7v^2+2v^3\Big]w\\
&\quad +Z(G_{LKL},q-s,v)
\end{aligned} \tag{11.30}$$

where $Z(G_{LKL},q,v)=Z(G_{KLL},q,v)$ was given above in (11.28). For the G_{KLL} graph we calculate

$$\begin{aligned}
Z(G_{KLL},q,s,v,w) &= Z(G_{KLL},s,v)w^5\\
&\quad +s(q-s)\Big[5s^3+15s^2v+16sv^2+2sv^3+5v^3+v^4\Big]w^4\\
&\quad +s(q-s)\Big[-10s^3+10s^2(q-v)+sv(15q+2v-v^2)\\
&\quad +6qv^2+qv^3+4v^3+v^4\Big]w^3+s(q-s)\Big[-10(q-s)^3\\
&\quad +10(q-s)^2(q-v)+(q-s)v(15q+2v-v^2)\\
&\quad +6qv^2+qv^3+4v^3+v^4\Big]w^2+s(q-s)\Big[5(q-s)^3+15(q-s)^2v\\
&\quad +16(q-s)v^2+2(q-s)v^3+5v^3+v^4\Big]w\\
&\quad +Z(G_{KLL},q-s,v).
\end{aligned} \tag{11.31}$$

Thus, $Z(G_{LKL}, q, s, v, w)$ is not, in general, equal to $Z(G_{KLL}, q, s, v, w)$ and, taking the $v = -1$ special case, $Ph(G_{LKL}, q, s, w)$ is not, in general, equal to $Ph(G_{KLL}, q, s, w)$. For the differences, we find

$$\begin{aligned} & Z(G_{LKL}, q, s, v, w) - Z(G_{KLL}, q, s, v, w) \\ & \quad = \mu \Big[s(w-1) + (1+w)v^2 + 2wv + q + 2v \Big] \end{aligned} \tag{11.32}$$

and thus

$$Ph(G_{LKL}, q, s, w) - Ph(G_{KLL}, q, s, w) = s(q-s)w(w-1)^2 \Big[s(w-1) + q - 1 - w \Big]. \tag{11.33}$$

These calculations provide another illustration of how $Z(G, q, s, v, w)$ can distinguish between different graphs that yield the same Tutte polynomial, and how $Ph(G, q, s, w)$ can distinguish between different graphs that yield the same chromatic polynomial.

In the context of graphs that can be decomposed into the union of subgraphs that intersect in a complete graph, it is also useful to calculate Z and Ph for the graph consisting of two K_3's meeting at a common vertex, $\bowtie$, denoted G_{KK}. This graph has $n = 5$, $e = 6$, and $c = 2$. The Tutte polynomial is $T(G_{KK}, x, y) = (x + x^2 + y)^2$, or equivalently

$$Z(G_{KK}, q, v) = \frac{Z(K_3, q, v)^2}{Z(K_1, q, v)} = q(q^2 + 3qv + 3v^2 + v^3)^2 \tag{11.34}$$

so that

$$P(G_{KK}, q) = \frac{P(K_3, q)^2}{P(K_1, q)} = q(q-1)^2(q-2)^2. \tag{11.35}$$

Note that these polynomials factorize. This is not the case with $Z(G_{KK}, q, s, v, w)$ and $Ph(G_{KK}, q, s, w)$. We calculate

$$\begin{aligned} Z(G_{KK}, q, s, v, w) = {} & Z(G_{KK}, s, v)w^5 + s(q-s)(s+v)(5s^2 + 13sv + 12v^2 + 4v^3)w^4 \\ & + 2s(q-s)\Big[-5s^3 + s^2(5q - 6v) + sv(9q - v^2) + 5qv^2 \\ & + (q+3)v^3 + v^4 \Big] w^3 + 2s(q-s)\Big[-5(q-s)^3 + (q-s)^2(5q-6v) \\ & + (q-s)v(9q - v^2) + 5qv^2 + (q+3)v^3 + v^4 \Big] w^2 \\ & + s(q-s)(q-s+v)\Big[5(q-s)^2 + 13(q-s)v + 12v^2 + 4v^3 \Big] w \\ & + Z(G_{KK}, q-s, v) \end{aligned} \tag{11.36}$$

and hence

$$\begin{aligned} Ph(G_{KK}, q, s, w) = {} & P(G_{KK}, s)w^5 + s(q-s)(s-1)^2(5s-8)w^4 \\ & + 2s(q-s)(s-1)\Big[5s(q-s) + s - 4q + 2 \Big] w^3 \end{aligned}$$

$$+2s(q-s)(q-s-1)\Big[5s(q-s)+(q-s)-4q+2\Big]w^2$$
$$+s(q-s)(q-s-1)^2\Big[5(q-s)-8\Big]w+P(G_{KK},q-s). \tag{11.37}$$

This example thus further illustrates how the factorization properties of the field-dependent Potts model partition function and set-weighted chromatic polynomial differ from those of the zero-field Potts model partition function and (unweighted) chromatic polynomial.

12 Conclusions

In this paper we have presented exact results on $Z(G,q,s,v,w)$, the partition function of the Potts model in an external generalized magnetic field that favors or disfavors spin values in a subset I_s of the full set I_q on various families of graphs G, and on $Ph(G,q,s,w)$, the weighted-set chromatic polynomial. In particular, we have presented new general calculations of $Z(G,q,s,v,w)$ for the case of path (line) graphs L_n and $Ph(Wh_n^{(p)},q,s,w)$ for p-wheel graphs $Wh_n^{(p)}$. We have discussed various features of our exact results for path, line, circuit, star, and complete graphs. We have derived powerful new upper and lower bounds on $Z(G,q,s,v,w)$ in terms of zero-field Potts partition functions with certain transformed arguments. We have also proved inequalities for the field-dependent Potts partition function on different families of tree graphs. An important property of $Z(G,q,s,v,w)$ is the fact that it can distinguish between Tutte-equivalent graphs, and similarly, $Ph(G,q,s,w)$ can distinguish between certain chromatically equivalent graphs. We have elucidated this property with our inequalities and with explicit calculations. Some new results for quantities such as $f(\{G\},q,s,v,w)$, $\Phi(\{G\},q,s,w)$, and q_c defined in the $n\to\infty$ limits of recursive graph families have also been given.

There are a number of interesting directions for future study. One direction is to investigate additional graph-theoretic applications of $Z(G,q,s,v,w)$ and $Ph(G,q,s,w)$. Second, it is clearly valuable to calculate $Z(G,q,s,v,w)$ and $Ph(G,q,s,w)$ for other families of graphs, in particular, recursive ones such as lattice strips and study their properties. We have described some of the interesting differences in thermodynamic behavior between our model with a generalized external magnetic field and the Potts model with a conventional magnetic field. It would be worthwhile to explore these differences further for various values of q, s, H, and temperature on lattices in $d\ge 2$ dimensions, using methods such as series expansions and Monte Carlo simulations.

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Appendix 1

In this appendix we present some explicit expressions for the set-weighted chromatic polynomials $Ph(Wh_n^{(p)},q,s,w)$ obtained from our general formula (7.12). We begin with case $p=1$, i.e., the family of wheel graphs, $Wh_n^{(1)}$. The first nondegenerate case is $n=4$, for which $Wh_4^{(1)}=K_4$, as noted (cf. (6.6)). For $n=5$, we have

$$Ph(Wh_5^{(1)},q,s,w)=s(s-1)(s-2)(s^2-5s+7)w^5$$

$$
\begin{aligned}
&+ s(s-1)[5s^2 - 19(s-1)](q-s)w^4 \\
&+ 2s(s-1)(q-s)(5qs - 7q - 5s^2 + 3s + 6)w^3 \\
&+ 2s(q-s)(q-s-1)(-4q + 5qs - 5s^2 + 6 - 3s)w^2 \\
&+ s(q-s)(q-s-1)[5(q-s)^2 - 19(q-s-1)]w \\
&+ (q-s)(q-s-1)(q-s-2)[(q-s)^2 - 5(q-s) + 7].
\end{aligned}
\tag{13.1}
$$

For $n = 6$,

$$
\begin{aligned}
Ph(Wh_6^{(1)}, q, s, w) = {} & s(s-1)(s-2)(s-3)(s^2 - 4s + 5)w^6 \\
& + 2s(s-1)(s-2)[3s^2 - 11(s-1)](q-s)w^5 \\
& + 5s(s-1)(q-s)(3qs^2 - 9qs + 7q - 3s^3 + 7s^2 - s - 5)w^4 \\
& + 20s(s-1)^2(q-s)(q-s-1)^2 w^3 \\
& + 5s(q-s)(q-s-1)(3q^2 s - 2q^2 + 6q - 6qs^2 - 5qs + 3s^3 \\
& + 7s^2 - 5 + s)w^2 + 2s(q-s)(q-s-1)(q-s-2)[(3(q-s)^2 \\
& - 11(q-s-1)]w + (q-s)(q-s-1)(q-s-2) \\
& \times (q-s-3)[(q-s)^2 - 4(q-s) + 5].
\end{aligned}
\tag{13.2}
$$

For the family of p-wheel graphs with $p = 2$, i.e., $Wh_n^{(2)}$, we first note that $Wh_4^{(2)} = K_2 + K_2 = K_4$ and $Wh_5^{(2)} = K_2 + K_3 = K_5$, both of which are subsumed by our general formula (6.2). For $Wh_6^{(2)}$ our general result (7.12) yields

$$
\begin{aligned}
Ph(Wh_6^{(2)}, q, s, w) = {} & s(s-1)(s-2)(s-3)(s^2 - 7s + 13)w^6 \\
& + 2s(s-1)(s-2)(3s^2 - 17s + 25)(q-s)w^5 \\
& + s(s-1)(q-s)(15qs^2 - 63qs + 67q - 15s^3 + 50s^2 \\
& - 12s - 59)w^4 + 4s(s-1)(q-s)(q-s-1)[5s(q-s) \\
& - 8q + 13]w^3 + s(q-s)(q-s-1)(15q^2 s - 13q^2 - 37qs \\
& + 55q - 30qs^2 + 12s + 15s^3 + 50s^2 - 59)w^2 \\
& + 2s(q-s)(q-s-1)(q-s-2)[3(q-s)^2 - 17(q-s) + 25]w \\
& + (q-s)(q-s-1)(q-s-2)(q-s-3)[(q-s)^2 \\
& - 7(q-s) + 13].
\end{aligned}
\tag{13.3}
$$

Explicit examples for higher values of p and n can be calculated in a similar manner from our general formula (7.12).

Appendix 2

As discussed in the text, an important property of $Z(G, q, s, v, w)$ is that it can distinguish between different Tutte-equivalent graphs, and a similarly important property of

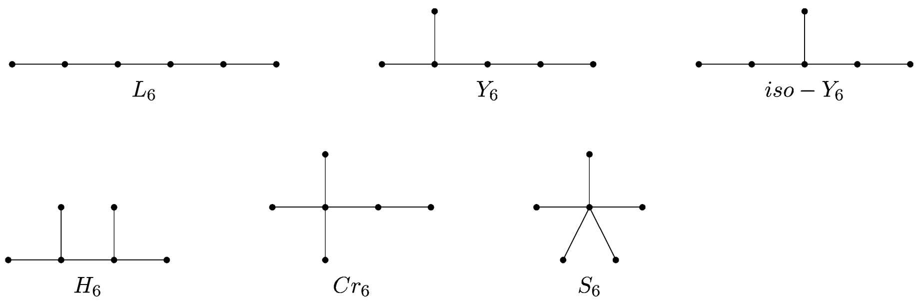

Fig. 1 Tree graphs with $n = 6$ vertices

$Ph(G,q,s,w)$ is that it can distinguish between many chromatically equivalent graphs. Tree graphs provide a basic context in which to explore this property, since any two tree graphs are both chromatically equivalent and Tutte-equivalent. In the present text and in earlier work we have given $Z(G,q,s,v,w)$ for tree graphs up to and including $n = 5$ vertices. Here we give results for tree graphs with $n = 6$ vertices. There are six such tree graphs, as shown in Fig. 6 of Ref. [3], reproduced here as Fig. 1 for the reader's convenience, labelled (i) L_n, (ii) Y_6, (iii) isoY_6, (iv) H_6, (v) Cr_6, and (vi) S_6. These are listed in order of increasing maximal vertex degree Δ. There are thus $\binom{6}{2} = 15$ differences of $Z(G,q,s,v,w)$ polynomials for these six graphs. We list these below, as differences with respect to the graph with highest maximal vertex degree, S_6 first, then with respect to the graph with next-highest maximal vertex degree, Cr_6, and so forth. Furthermore, since the arguments are the same for all of the partition functions, we omit them, writing $Z(G,q,s,v,w) - Z(H,q,s,v,w) \equiv Z(G) - Z(H)$. The differences are

$$Z(S_6) - Z(Cr_6) = \mu\Big[(3s^2 + 3sv + v^2)w^2 + \{6s(q-s) + 3qv + v^2\}w + \{3(q-s)^2 + 3(q-s)v + v^2\}\Big] \tag{14.1}$$

$$Z(S_6) - Z(H_6) = \mu[(2s + v)w + \{2(q-s) + v\}]^2 \tag{14.2}$$

$$Z(S_6) - Z(IsoY_6) = \mu\Big[(5s^2 + 6sv + 2v^2)w^2 + \{10s(q-s) + 5qv + 2v^2\}w + \{5(q-s)^2 + 6(q-s)v + 2v^2\}\Big] \tag{14.3}$$

$$Z(S_6) - Z(Y_6) = \mu\Big[(5s^2 + 6sv + 2v^2)w^2 + \{10s(q-s) + 6qv + 3v^2\}w + \{5(q-s)^2 + 6(q-s)v + 2v^2\}\Big] \tag{14.4}$$

$$Z(S_6) - Z(L_6) = \mu\Big[(6s^2 + 8sv + 3v^2)w^2 + \{12s(q-s)$$

$$+7qv+4v^2\}w+\{6(q-s)^2+8(q-s)v+3v^2\}\Big] \tag{14.5}$$

$$\begin{aligned} Z(Cr_6)-Z(H_6) &= \mu\Big[q+s(w-1)+v\Big]\Big[q+s(w-1)+vw\Big] \\ &= \mu\Big[s(s+v)w^2+\{2s(q-s)+v(q+v)\}w \\ &\quad +(q-s)(q-s+v)\Big] \end{aligned} \tag{14.6}$$

$$\begin{aligned} Z(Cr_6)-Z(IsoY_6) &= \mu\Big[(s+v)(2s+v)w^2+(2s+v)\{2(q-s)+v\}w \\ &\quad +(q-s+v)\{2(q-s)+v\}\Big] \end{aligned} \tag{14.7}$$

$$\begin{aligned} Z(Cr_6)-Z(Y_6) &= \mu\Big[2\{q+s(w-1)\}+v(w+1)\Big] \\ &\quad \times\Big[q+s(w-1)+v(w+1)\Big] \\ &= \mu\Big[(s+v)(2s+v)w^2+\{4s(q-s)+3qv+2v^2\}w \\ &\quad +(q-s+v)\{2(q-s)+v\}\Big] \end{aligned} \tag{14.8}$$

$$\begin{aligned} Z(Cr_6)-Z(L_6) &= \mu\Big[(s+v)(3s+2v)w^2+\{6s(q-s)+4qv+3v^2\}w \\ &\quad +(q-s+v)\{3(q-s)+2v\}\Big] \end{aligned} \tag{14.9}$$

$$\begin{aligned} Z(H_6)-Z(IsoY_6) &= \mu\Big[(s+v)^2w^2+\{2s(q-s)+qv\}w \\ &\quad +(q-s+v)^2\Big] \end{aligned} \tag{14.10}$$

$$\begin{aligned} Z(H_6)-Z(Y_6) &= \mu\Big[(s+v)^2w^2+\{2s(q-s)+2qv+v^2\}w \\ &\quad +(q-s+v)^2\Big] \end{aligned} \tag{14.11}$$

$$\begin{aligned} Z(H_6)-Z(L_6) &= \mu\Big[2(s+v)^2w^2+\{4s(q-s)+3qv+2v^2\}w \\ &\quad +2(q-s+v)^2\Big] \end{aligned} \tag{14.12}$$

$$Z(IsoY_6)-Z(Y_6)=\mu wv(q+v)=s(q-s)v^3(q+v)[w(w-1)]^2 \tag{14.13}$$

$$Z(IsoY_6)-Z(L_6)=\mu\Big[q+s(w-1)+v(w+1)\Big]^2$$

$$= \mu\Big[(s+v)w + q - s + v\Big]^2 \tag{14.14}$$

$$Z(Y_6) - Z(L_6) = \mu\Big[(s+v)^2 w^2 + \{2s(q-s) + v(q+v)\}w + (q-s+v)^2\Big] \tag{14.15}$$

In addition to our theorems (11.21) and (11.23), we observe that (for $0 \le s \le q$ and $w \ge 0$) all of these differences are non-negative for the ferromagnetic range $v \ge 0$.

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